NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4343

A COMPARISON OF TWO METHODS FOR CALCULATING TRANSIENT TEMPERATURES FOR THICK WALLS

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SUMMARY

A comparison is made of two different methods of calculating transient temperatures for thick walls with arbitrary variation of heat-transfer coefficient and adiabatic-wall temperature. Although numerical calculations for special cases for which the exact solutions are available show that both methods give satisfactory results, Hill's method (NACA Technical Note 4105) consistently gives nearly exact results with considerably less computing time, except for the case in which a temperature profile through the thick skin is desired. For this case, Dusinberre's method (Trans. A.S.M.E., vol. 67, no. 8) is much faster, though less accurate.

INTRODUCTION

No exact analytical method is available for computing the transient temperature for the general case of thick walls. Various finite-difference methods have therefore been proposed and used to compute transient wall temperatures. With the increased importance of high temperature in aircraft structural design, an evaluation of the merits of representative methods is warranted.

The existence of several other methods is fully acknowledged and no comprehensive comparison of all available methods is intended. It is intended merely to select two representative methods and to compare their results and computing times. The methods of Hill (ref. 1) and Dusinberre (ref. 2) have been selected for this purpose. In this paper only the basic one-dimensional case has been considered. Hand calculations with a desk computer were made rather than resorting to an electronic computer, because in many engineering applications fast and direct answers are required, and in some problems, programing time on a computer becomes excessive.

The first application of the finite-difference method for determining thick-wall temperatures is credited to Schmidt (ref. 3). This method employs a ratio of incremental time to increment of distance into the

wall that is fixed by the material properties of the wall. Dusinberre has introduced an extension of Schmidt's method whereby the ratio of time increment to distance increment can be varied to introduce smaller time steps if desired. An increase of accuracy relative to Schmidt's method is thereby possible. Dusinberre's method includes Schmidt's method and can be reduced to it by the adjustment of a coefficient.

Hill's method represents a considerably different approach to the thick-wall problem. Finite differences are taken only in the time variable, the equations used being already integrated with respect to distance.

Following a description of both methods, sample problems are solved. Problems permitting exact solutions were chosen to make possible an evaluation of the accuracy of the methods. A time study of the methods was also made to determine the relative labor involved. An attempt is made to point out the areas of application wherein one method might be more advantageous than the other.

It should be mentioned that Hill's method allows the outer-surface temperature to be readily determined if the time history of the inner-surface temperature is known. This fact makes Hill's method extremely advantageous to investigators in the field of aerothermodynamics where, generally, thermocouples are mounted on the inside surfaces of specimens and the heating rates and outer-surface temperatures are desired. The calculation of the outer-surface temperatures from the inner-surface temperatures can also be made with Dusinberre's method, but the process is much more laborious and less straightforward.

SYMBOLS

c	specific heat, Btu/(lb)($^{\circ}$F)
F	coefficient in Dusinberre's method
G	heat-capacity parameter, $\rho c l$, Btu/(sq ft)($^{\circ}$F)
H	heating-rate parameter, $h\delta\pi^2/16G$
h	heat-transfer coefficient, Btu/(hr)(sq ft)($^{\circ}$F)
k	diffusivity, $K/c\rho$, (sq ft)/hr
K	conductivity, (Btu)(ft)/(hr)(sq ft)($^{\circ}$F)
l	wall thickness, ft

M	memory coefficient in Hill's method
m	step number
$P = c\rho(\Delta x)^2/K\delta$	
$Q = h(\Delta x)/K$	
r	radiation rate, Btu/(hr)(sq ft)
R	radiation-rate parameter, $r\delta\pi^2/16G$
t	time from start of heating, hr
T	temperature, ^{o}R
δ	time interval, hr
ρ	weight density, lb/cu ft
θ	memory coefficient
Δx	distance increment, ft

Subscripts:

aw	adiabatic wall
i	inner
j	distance increment number
m	step number

OUTLINE OF PROBLEM AND METHODS

The two methods were used to calculate the inner- and outer-surface temperatures of a thermally thick plane copper wall. Wall thicknesses of 1/2 inch, 1 inch, and 3 inches were used. The walls were assumed to be insulated at the inner surface and their thermal properties were assumed to be constant. The given input function was a time history of adiabatic-wall temperature and heat-transfer coefficient.

Hill's Method

Reference 1 gives a complete discussion and derivation of the equations for Hill's method of computing transient temperatures of thick walls for any arbitrary variation of adiabatic-wall temperature and heat-transfer coefficient. The final equations, as presented in reference 1, are given here for convenience.

Outer-surface temperature.- The outer-surface temperature at the time $m\delta$ is given by

$$T_m = \frac{(HT_{aw})_m + (HT_{aw} - HT)_{m-1} - M_2T_{m-1} - M_3T_{m-2} - \cdot\ \cdot\ \cdot - M_mT_1 - R_m - R_{m-1}}{M_1 + H_m} \tag{1}$$

where

$$H = \frac{h\delta\pi^2}{16G} \tag{2}$$

$$R = \frac{r\delta\pi^2}{16G} \tag{3}$$

$$G = \rho c l \tag{4}$$

For example,

$$T_1 = \frac{(HT_{aw})_1 + (HT_{aw})_0 - R_1 - R_0}{M_1 + H_1} \tag{5}$$

$$T_2 = \frac{(HT_{aw})_2 + (HT_{aw} - HT)_1 - M_2T_1 - R_2 - R_1}{M_1 + H_2} \tag{6}$$

Values of the memory coefficients M are taken from table I (most of which is reproduced from ref. 1). Interpolation in table I is avoided by working with a time increment δ that results in a value of $k\delta/l^2$ listed in the table.

Inner-surface temperature.- The inner-surface temperature at the time $m\delta$ is given by

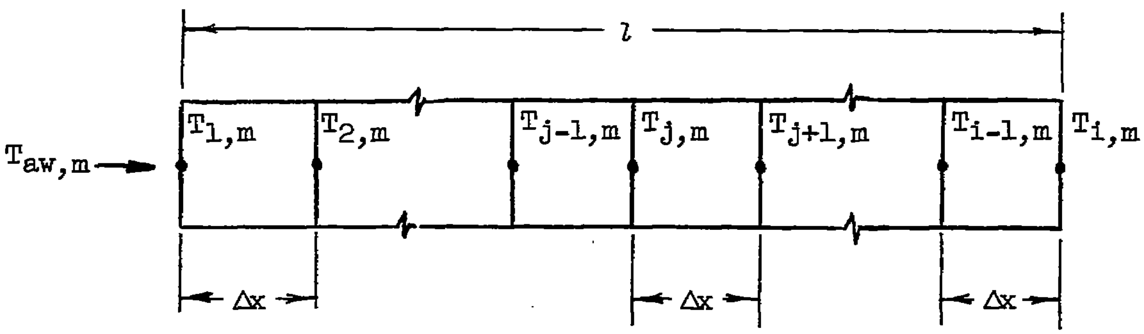
$$T_{i,m} = T_m - \left(\theta_1T_m + \theta_2T_{m-1} + \cdot\ \cdot\ \cdot + \theta_mT_1\right) \tag{7}$$

Values of θ are taken from table I.

Dusinberre's Method

Dusinberre's method essentially consists of dividing the wall into a finite number of slabs and taking a heat balance for each slab. The relations used in calculating the transient-temperature time histories by this method are taken from reference 2 and repeated in this section.

A remark to clarify the subscript notation is in order. The first subscript on the temperature denotes the block for which the temperature is being calculated and the second subscript is the time at which the temperature is being calculated. On the averaging coefficients (the F-coefficients) the first subscript denotes the temperature used in the averaging process and the second subscript shows the block for which the temperature is being calculated. Subscript i is the inner surface and subscript j is any intermediate block. The following sketch shows this notation:



Outer-surface temperature.- For the outer surface,

$$T_{1,m} = F_{aw,1}T_{aw,m-1} + F_{1,1}T_{1,m-1} + F_{2,1}T_{2,m-1} \tag{8}$$

where

$$F_{aw,1} = \frac{2Q}{P} = h\,\frac{\Delta x}{K}\,\frac{2}{P} \tag{9}$$

$$F_{2,1} = \frac{2}{P} \tag{10}$$

$$F_{1,1} = 1 - F_{aw,1} - F_{2,1} \tag{11}$$

$$Q = \frac{h\,\Delta x}{K} \tag{12}$$

and P is some number such that

$$P \geqq 2 + 2Q \tag{13}$$

The parameter P can have any value which satisfies equation (13), the size of P chosen dictating the time interval used, as shown by the relation

$$\delta = \frac{c\rho(\Delta x)^2}{KP} \tag{14}$$

Equations (13) and (14) impose a maximum time increment for a given Δx but there is no limit to the minimum value of time increment which can be used by increasing the parameter P. Obviously, a time interval could be chosen and a value of P calculated from equation (14). If the P selected is large enough to be greater than $2 + 2Q$ for all values of heat-transfer coefficient, a constant δ can be used which makes computing considerably faster and easier, the reason for this being that only $F_{aw,1}$ and $F_{1,1}$ have to be computed for each time interval. The other F-coefficients are constant throughout the problem.

Inner-surface temperature.- For the inner surface,

$$T_{i,m} = F_{i-1,i}T_{i-1,m-1} + F_{i,i}T_{i,m-1} \tag{15}$$

where the subscript i is the number of the last cube, and

$$F_{i-1,i} = \frac{2}{P} \tag{16}$$

$$F_{i,i} = 1 - F_{i-1,i} \tag{17}$$

Intermediate wall temperatures.- The following equation permits a calculation of temperatures in the interior of the wall:

$$T_{j,m} = \frac{T_{j-1,m-1} + (M - 2)T_{j,m-1} + T_{j+1,m-1}}{M} \tag{18}$$

Use of this equation allows temperature profiles through the slab to be calculated without any difficulty. Indeed, it is a consequence of this method that the entire temperature profile must be calculated, because the temperature at any point in the slab depends on the temperatures on either side of it at the preceding time interval as well as its own temperature at the preceding time interval. This may be advantageous

because Hill's method uses a series of cosine terms for the calculation of internal temperature profiles, which makes this calculation considerably more awkward.

Numerical Calculations

In the numerical calculations special cases were chosen for which exact analytical answers could be obtained, in order that the accuracy of each method might be determined.

The transient-temperature histories of thick copper walls were computed by both methods. The following thermal properties were assumed:

$$K = 227\ (\mathrm{Btu})(\mathrm{ft})/(\mathrm{hr})(\mathrm{sq\ ft})(^{\circ}\mathrm{F})$$

$$c = 0.09192\ \mathrm{Btu}/(\mathrm{lb})(^{\circ}\mathrm{F})$$

$$\rho = 560\ \mathrm{lb}/(\mathrm{cu\ ft})$$

$$k = K/c\rho = 4.41\ (\mathrm{sq\ ft})/\mathrm{hr}$$

Two different heating cases were considered.

Case I.- The heat-transfer coefficient h was assumed to be 100 Btu/(hr)(sq ft)($^{\circ}$F) and to be held constant. The adiabatic-wall temperature was assumed to increase linearly with time, from a value of 0° F at zero time to a value of 10,000° F at 10 seconds. The transient temperatures of 1/2-, 1-, and 3-inch-thick copper walls were computed. The outer- and inner-surface temperatures for case I are given in figures 1 to 3.

Case II.- The heat-transfer coefficient and adiabatic-wall temperature were assumed to have an arbitrary variation for the 1/2-inch and the 3-inch copper wall. Values of h are the same for both examples, whereas T_{aw} is slightly different. The values of h and T_{aw} used in this case were obtained as follows: An outer-surface temperature was assumed and the heat input required to give this temperature was calculated by an exact analytical method. This heat input was then used to determine values of h and T_{aw}. This was done so that a solution with h variable could be obtained from an analytical method. The temperatures calculated by the methods of Hill and Dusinberre were then compared with the original assumed temperatures. The values of h and T_{aw} used are given in the following table:

Time, sec	T_{aw}, °F, for -		h, $\frac{\text{Btu}}{(\text{hr})(\text{sq ft})(^{\circ}\text{F})}$
	1/2-in. wall	3-in. wall	
0	0	0	36.0
1	2,485	2,484	45.0
2	4,094	4,088	52.2
3	4,932	4,915	60.0
4	5,263	5,227	66.6
5	5,387	5,325	69.0
6	5,356	5,265	66.6
7	5,107	4,986	60.0
8	4,335	4,184	52.2
9	2,769	2,591	45.0
10	297.5	100	36.0

The results for case II are shown in figures 4 and 5.

Computations.- For Dusinberre's method a choice must be made of the size of distance increments into which the wall is divided. One difficulty of the Schmidt or Dusinberre method for relatively thin walls is the small time step required by equation (14), and therefore computing times which are long relative to the time required for thicker walls. To keep the computing times within reason, the 1/2-inch wall was divided into only two increments. The increments selected for the different wall thicknesses are as follows:

Wall thickness, in.	Number of increments
1/2	2
1	2
3	6

Hill's method uses the whole wall thickness, and thus removes the necessity of choosing an incremental thickness.

RESULTS AND DISCUSSION

In case I, wherein the adiabatic-wall temperature increases from 0° F to $10{,}000^{\circ}$ F in 10 seconds, typical results were as follows:

Case I

l, in.	Dusinberre				Hill			
	δ, sec	Maximum error		Comp. time, min	δ, sec	Maximum error		Comp. time, min
		$^\circ$F	Percent max. temp.			$^\circ$F	Percent max. temp.	
1/2	0.15	11	< 2	183	1.4172 .7086	3 1	< 1 < 1	28 50
1	.5 .25	25 18	6 4	62 103	1.131	1	< 1	26
3	.6	15	4	98	1.020	2	< 1	32

For the 1/2-inch wall the error with Hill's method is about 1/4 and the computing time about 1/5 that of Dusinberre's method. For the 3-inch wall the error is about 1/8 and the time is about 1/3 that for Dusinberre's method. In general, the maximum time interval was used in computing by Dusinberre's method. However, it has been found that a smaller δ improves the accuracy, as shown in figure 2. Since the value of P used for Dusinberre's method was close to 2, the results for Schmidt's method would be similar. As shown by the results from alternate time increments, since Hill's method gives accurate results from substantially fewer steps, it is a waste of time to use very fine steps. For the thermally thinner walls - for example the 1/2-inch wall - the small δ required to satisfy the Dusinberre (or Schmidt) relations necessitates a long computation time.

In case II, wherein the adiabatic-wall temperature both rose and fell in a 10-second period while the heat-transfer coefficient varied also, the results were:

Case II

l, in.	Dusinberre				Hill			
	δ, sec	Maximum error		Computing time, min	δ, sec	Maximum error		Computing time, min
		$^\circ$F	Percent max. temp.			$^\circ$F	Percent max. temp.	
1/2	0.15	2	< 1	128	0.71	0.5	< 1	60
3	.6	3	2	57	1.020	.5	< 1	34

In this case the maximum error in the Dusinberre method was only 2° F for the 1/2-inch wall. However, for an equal error, Hill's method required only 1/5 as long. For the 3-inch wall the times were about twice as large for Dusinberre's as for Hill's method. On the inner surface the error in Dusinberre's method, while small, was a substantial percentage of the rise of the inner-surface temperature.

Dusinberre's method is well suited to obtaining the temperature distribution through the wall since in all cases the distribution is obtained as a necessary consequence of the computation. Such distributions are shown for case I for the 3-inch-thick wall in figure 6. At 6 seconds a temperature distribution computed from an exact formula from reference 1 is shown for comparison. The maximum error is 12° F or about 7 percent. It is also possible to compute the temperature distribution by Hill's method. This was done by the equation of appendix C of reference 1. The results are in almost perfect agreement with the exact theory.

CONCLUDING REMARKS

From the examples presented herein, as well as from other examples presented in reference 1, it appears that for any reasonable step size Hill's method is, practically, an exact method. On the other hand, Dusinberre's method, with reasonable step sizes, gives a good approximation. The same statement applies to the Schmidt method. Hill's method is also substantially faster than Dusinberre's method (or Schmidt's method) if only the two surface temperatures are required. If temperature distributions through the wall are required, Hill's method is slower but practically exact. Either method is suitable for machine calculations. Exact (classical) methods are not available except for special cases.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., May 16, 1958.

REFERENCES

1. Hill, P. R.: A Method of Computing the Transient Temperature of Thick Walls From Arbitrary Variation of Adiabatic-Wall Temperature and Heat-Transfer Coefficient. NACA TN 4105, 1957.

2. Dusinberre, G. M.: Numerical Methods for Transient Heat Flow. Trans. A.S.M.E., vol. 67, no. 8, Nov. 1945, pp. 703-712.

3. Schmidt, Ernst: Thermodynamics. The Clarendon Press (Oxford), 1949.

TABLE I.- VALUES OF MEMORY COEFFICIENTS

(a) Values of M

$k\delta/l^2$ / m	0.01	0.02	0.05	0.1	0.2	0.5	1.0	2.0	5.0
1	0.09281491	0.13125147	0.20752122	0.29347746	0.41495581	0.64728217	0.85683729	1.02954122	-1.15145422
2	-.01594178	-.02252780	-.03560872	-.05052075	-.07836600	-.22817193	-.51142943	-.82682879	-1.06920825
3	-.02682329	-.03794254	-.06002358	-.08722726	-.14861514	-.30052435	-.31662004	-.20127594	-.08224562
4	-.00957530	-.01354153	-.02166582	-.03696546	-.07338883	-.08405219	-.02634648	-.00142616	-.00000036
5	-.00553400	-.00782681	-.01320028	-.02628367	-.04463344	-.02447704	-.00223431	-.00001026	-.00000000
6	-.00373390	-.00528488	-.00983262	-.02025561	-.02724662	-.00712803	-.00018948	-.00000007	
7	-.00274116	-.00389295	-.00809228	-.01579610	-.01663398	-.00207577	-.00001607		
8	-.00212389	-.00304410	-.00695594	-.01233887	-.01015501	-.00060449	-.00000136		
9	-.00170888	-.00249269	-.00608368	-.00964054	-.00619962	-.00017604	-.00000012		
10	-.00141415	-.00211866	-.00535620	-.00753254	-.00378485	-.00005126	-.00000001		
11	-.00119613	-.00185531	-.00472749	-.00588550	-.00231065	-.00001493			
12	-.00103005	-.00166301	-.00417648	-.00459860	-.00141064	-.00000435			
13	-.00090099	-.00151704	-.00369098	-.00359308	-.00086120	-.00000127			
14	-.00079666	-.00140173	-.00326234	-.00280743	-.00052576	-.00000037			
15	-.00071651	-.00130718	-.00288362	-.00219357	-.00032097	-.00000011			
16	-.00064769	-.00122688	-.00254890	-.00171393	-.00019595	-.00000003			
17	-.00059289	-.00115664	-.00225306	-.00133917	-.00011963	-.00000001			
18	-.00054714	-.00109385	-.00199156	-.00104635	-.00007303				
19	-.00050882	-.00103656	-.00176041	-.00081756	-.00004459				
20	-.00047652	-.00098373	-.00155610	-.00063880	-.00002722				
21	-.00044898	-.00093449	-.00137549	-.00049912	-.00001662				
22	-.00042534	-.00088826	-.00121584	-.00038998	-.00001015				
23	-.00040474	-.00084473	-.00107473	-.00030471	-.00000619				
24	-.00038678	-.00080357	-.00094999	-.00023809	-.00000378				
25	-.00037081	-.00076453	-.00083973	-.00018602	-.00000231				
26	-.00035655	-.00072753	-.00074227	-.00014535	-.00000141				
27	-.00034370	-.00069238	-.00065612	-.00011357	-.00000086				
28	-.00033195	-.00065894	-.00057997	-.00008874	-.00000052				
29	-.00032118	-.00062716	-.00051265	-.00006933	-.00000032				
30	-.00031123	-.00059693	-.00045315	-.00005417	-.00000020				

(b) Values of θ

$k\delta/l^2$ \ m	0.01	0.02	0.05	0.1	0.2	0.5	1.0	2.0	5.0
1	1.00000000	0.99999990	0.99956261	0.98873107	0.92596579	0.69945338	0.45623848	0.24814437	0.09999955
2	-.00000019	-.00019242	-.02166158	-.12553131	-.31355030	-.48642965	-.41618822	-.24630212	-.09999908
3	-.00002493	-.00333067	-.06963678	-.17636952	-.23787104	-.15098863	-.03665380	-.00182900	-.00000045
4	-.00033480	-.01195026	-.09012900	-.14882949	-.14587796	-.04396967	-.00310842	-.00001315	-.00000000
5	-.00144190	-.02167827	-.08988005	-.11749067	-.08906596	-.01280454	-.00026361	-.00000009	
6	-.00344274	-.02904777	-.08284990	-.09193126	-.05437468	-.00372885	-.00002236	-.00000000	
7	-.00595604	-.03361672	-.07435643	-.07184409	-.03319564	-.00108589	-.00000190		
8	-.00854572	-.03599626	-.06609621	-.05613647	-.02026587	-.00031622	-.00000016		
9	-.01092536	-.03687745	-.05854671	-.04386205	-.01237227	-.00009209	-.00000001		
10	-.01296009	-.03679326	-.05179171	-.03427135	-.00755325	-.00002682	-.00000000		
11	-.01461454	-.03611078	-.04579381	-.02677771	-.00461124	-.00000781			
12	-.01590635	-.03507182	-.04048318	-.02092260	-.00281516	-.00000227			
13	-.01687671	-.03383231	-.03578599	-.01634774	-.00171865	-.00000066			
14	-.01757368	-.03249148	-.03163302	-.01277320	-.00104923	-.00000019			
15	-.01804424	-.03111188	-.02796174	-.00998026	-.00064055	-.00000006			
16	-.01833036	-.02973243	-.02471644	-.00779802	-.00039106	-.00000002			
17	-.01846822	-.02837688	-.02184780	-.00609293	-.00023874	-.00000000			
18	-.01848810	-.02705930	-.01931206	-.00476067	-.00014575				
19	-.01841482	-.02578770	-.01707064	-.00371972	-.00008898				
20	-.01826876	-.02456606	-.01508936	-.00290638	-.00005432				
21	-.01806620	-.02339609	-.01333804	-.00227088	-.00003316				
22	-.01782041	-.02227782	-.01178998	-.00177434	-.00002025				
23	-.01754197	-.02121042	-.01042159	-.00138637	-.00001236				
24	-.01723931	-.02019254	-.00921203	-.00108323	-.00000755				
25	-.01691916	-.01922244	-.00814285	-.00084638	-.00000460				
26	-.01658696	-.01829829	-.00719776	-.00066131	-.00000281				
27	-.01624686	-.01741809	-.00636236	-.00051671	-.00000172				
28	-.01590228	-.01658000	-.00562392	-.00040373	-.00000105				
29	-.01555588	-.01578204	-.00497119	-.00031545	-.00000064				
30	-.01520973	-.01502235	-.00439422	-.00024648	-.00000039				

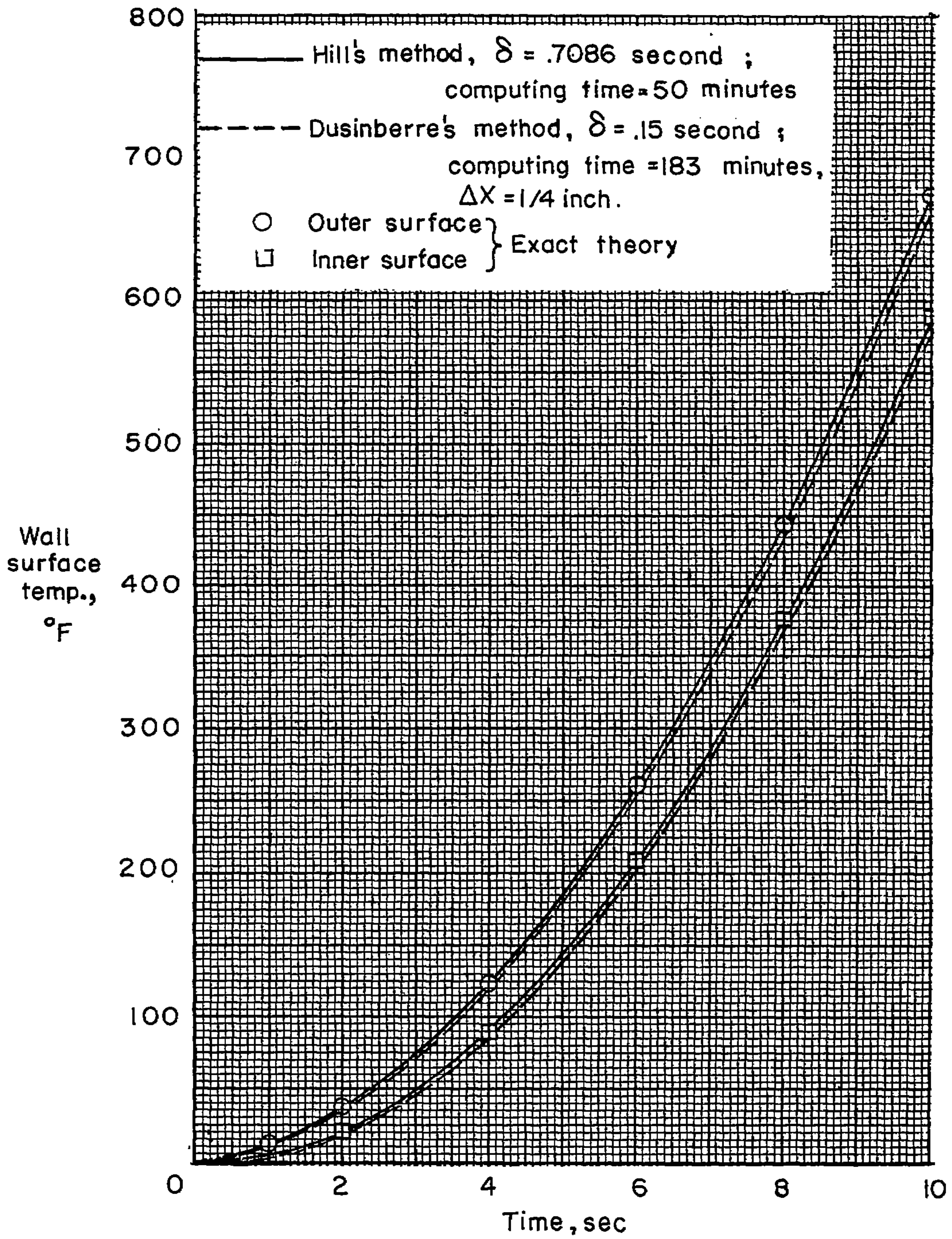


Figure 1.- Case I. Temperatures of 1/2-inch copper wall surfaces. Adiabatic-wall temperature varies linearly from 0° to 10,000° in 10 seconds; $h = 100$ Btu/(hr)(sq ft)(°F).

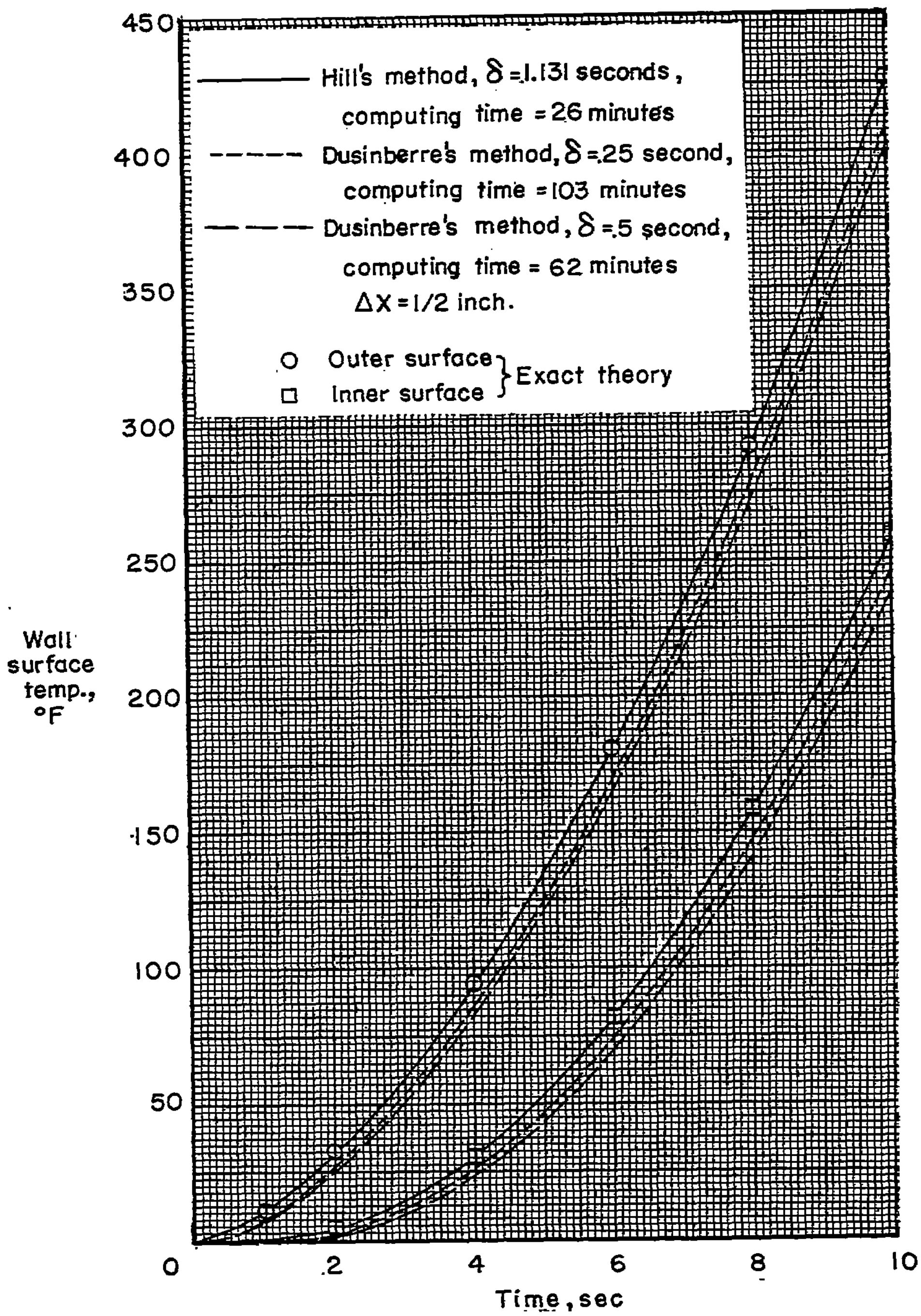


Figure 2.- Case I. Temperatures of 1-inch copper wall surfaces. Adiabatic-wall temperature varies linearly from 0° to 10,000° in 10 seconds; h = 100 Btu/(hr)(sq ft)(°F).

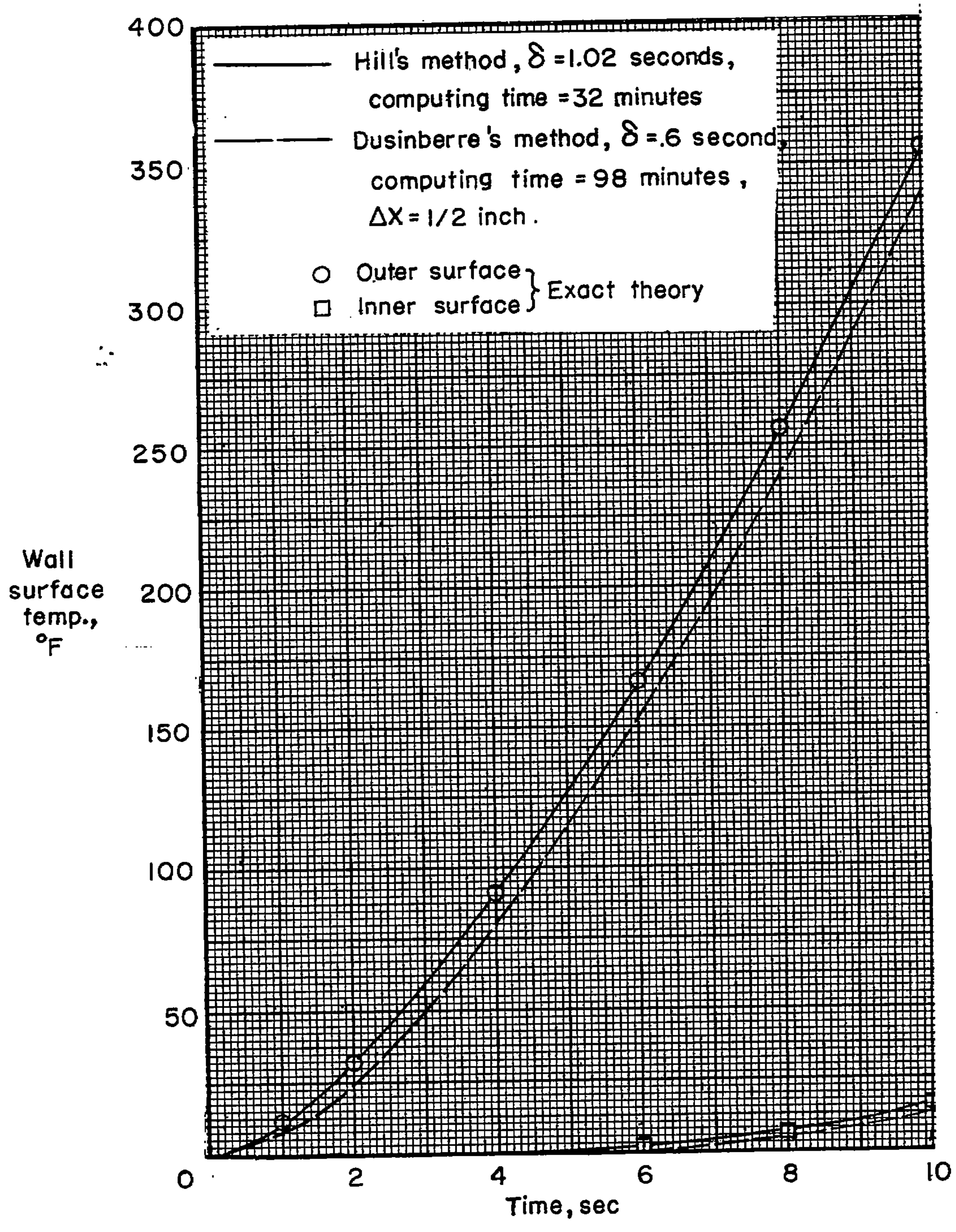


Figure 3.- Case I. Temperatures of 3-inch copper wall surfaces. Adiabatic-wall temperature varies linearly from 0° to 10,000° in 10 seconds; h = 100 Btu/(hr)(sq ft)(°F).

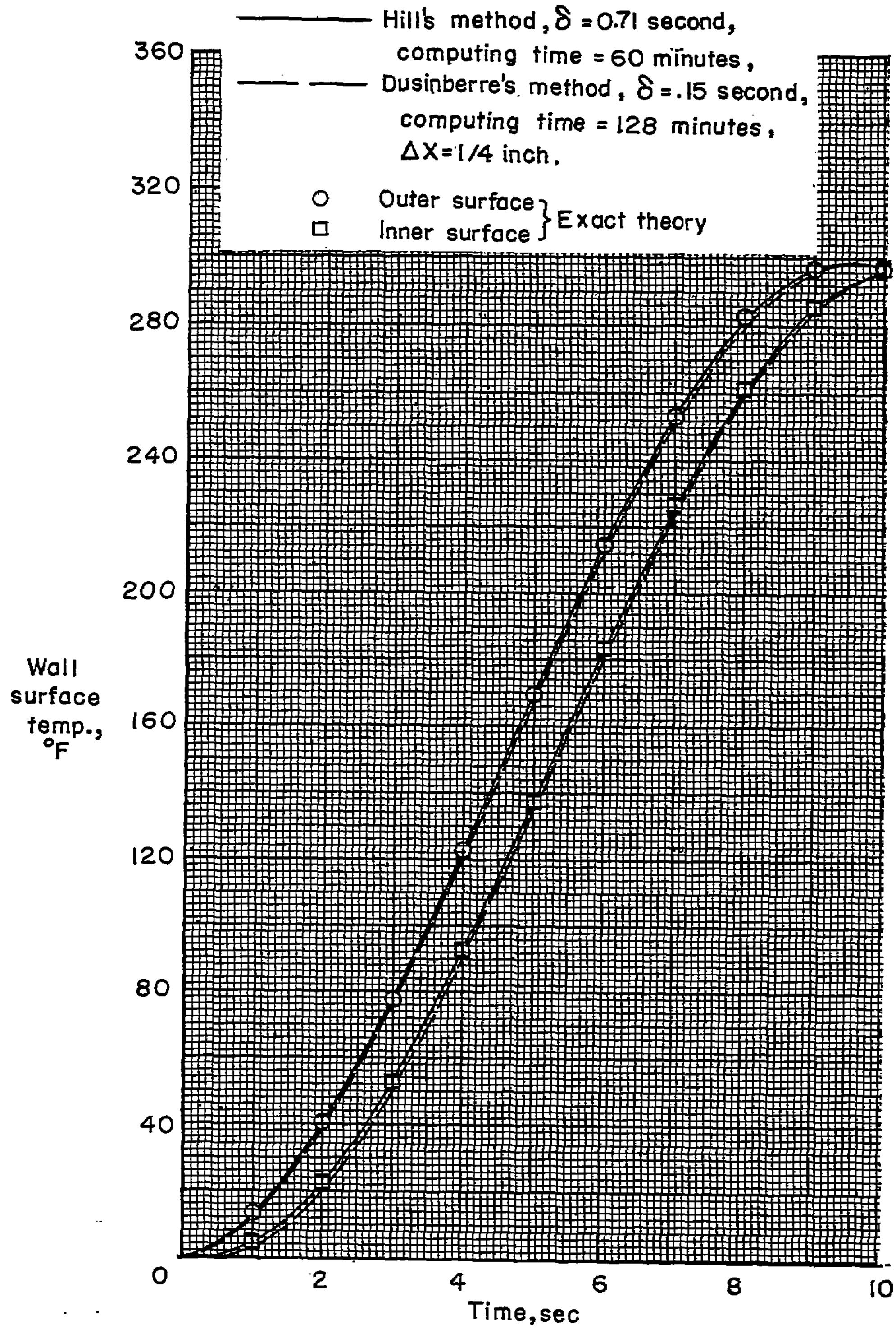


Figure 4.- Case II. Temperatures of 1/2-inch copper wall heated according to assigned history of h and T_{aw}.

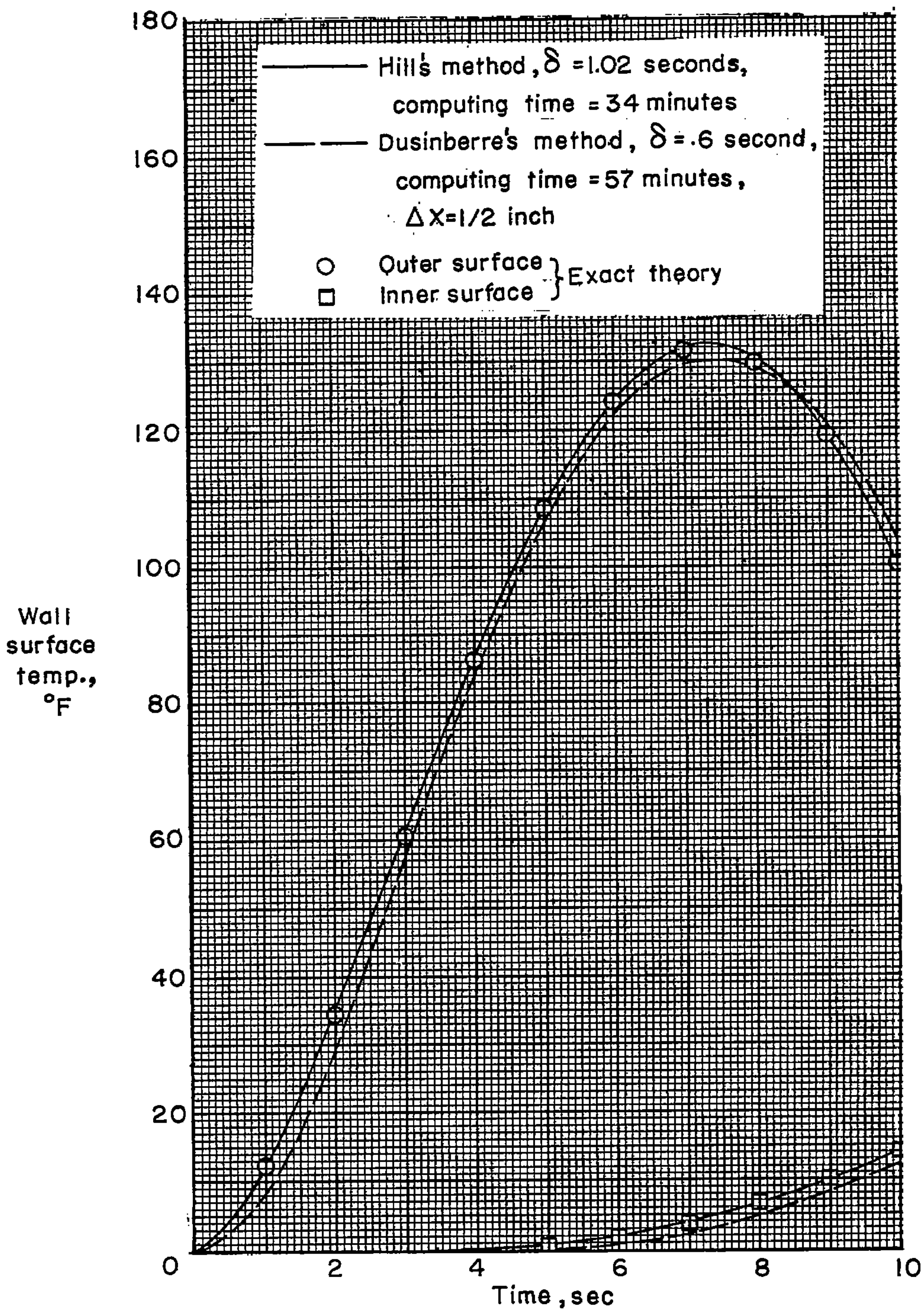


Figure 5.- Case II. Temperatures of 3-inch copper wall heated according to assigned history of h and T_{aw}.

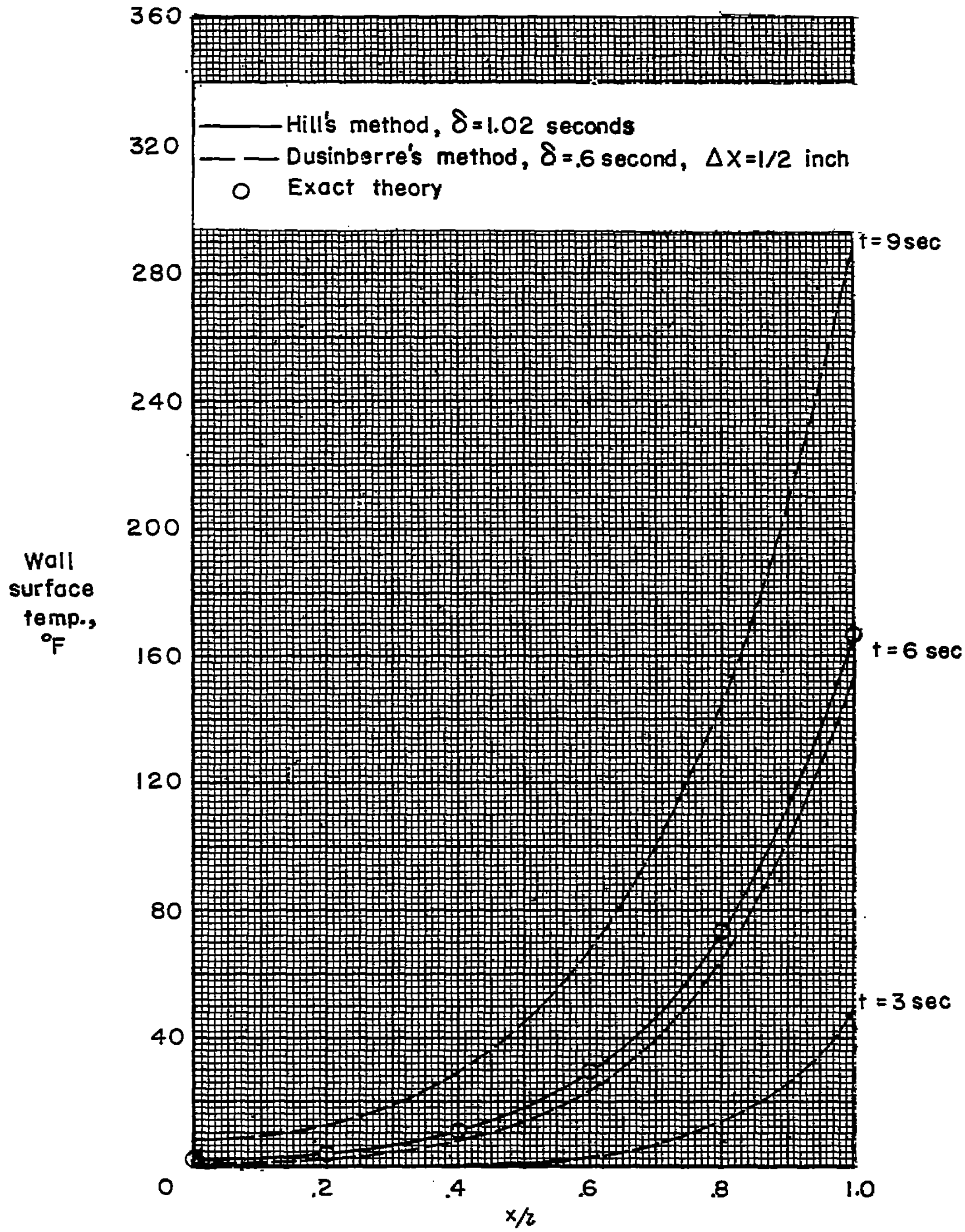


Figure 6.- Case I. Temperature profiles through 3-inch copper wall. Adiabatic-wall temperature varies linearly from 0° to 10,000° in 10 seconds; h = 100 Btu/(hr)(sq ft)(°F).